The Halting Problem
or what computers can’t do

Julian Bradfield
School of Informatics, University of Edinburgh
The Entscheidungsproblem

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*Write a computer program that can tell us whether any mathematical statement is true or false.*

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“For positive integers $n, a, b, c$, if $n > 2$, then $a^n + b^n \neq c^n$”

\[\downarrow\]

\textit{deciding machine}

\[\downarrow\]

“True!”
The end of optimism

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In the early 1930s, three people shook the foundations of mathematics.

Kurt Gödel  Alonzo Church  Alan Turing
Born in London in 1912, to parents who lived in India (his father was a civil servant). Fairly miserable time at school (Sherborne), where he was considered to be wasting his time being interested in science.
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By 1933, he had heard of Gödel’s shattering result, that no logic could prove all theorems.
In 1935, he heard about the Entscheidungsproblem.
What is computing?

Turing had to make precise Hilbert’s idea of ‘procedure’.
He gave a theoretical pencil and paper method of calculation; any modern language will do just as well.
He argued that if we can compute something, we can compute it with one of these programs.
Computing with computers

The second idea – based on Gödel’s idea in logic – was that a procedure can calculate things about a procedure.

To us, this is natural: you write a program, and your program is compiled or interpreted by another program, and eventually machine code runs on a real physical computer.

Then, it was not so natural.
Impossible computations

The third idea – again, based on Gödel’s technique in logic – was to find a statement about procedures that could not possibly be calculated by a procedure.

Your PC ran into a problem that it couldn't handle
The Halting Problem

The statement is: given a procedure, does it eventually finish computing, or does it get stuck in a loop?

\[ P, \text{ the halting decider} \]

\[ \text{“Good!”} \]
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“Bad!”
The Loop Snooper

Write a program that runs a procedure *on itself* and halts if it loops, and loops if it halts!
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\[ \downarrow \]

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The Loop Snooper

Write a program that runs a procedure *on itself* and halts if it loops, and loops if it halts!

![Diagram](image)

Q, the loop snooper

P, the halting decider

Say's 'Bad!'
The Loop Snooper

Write a program that runs a procedure on itself and halts if it loops, and loops if it halts!
The Loop Snooper Scoops the Halting Decider

Ask the Loop Snooper to analyse itself! What happens?

\[
\begin{align*}
Q, \text{ the loop snooper} & \quad A \rightarrow A \\
\downarrow & \\
P, \text{ the halting decider} & \\
\quad \text{Says ‘Bad!’} & \quad \text{Says ‘Good!’} \\
\quad \text{halt} & \quad \text{loop for ever} \\
\end{align*}
\]
The Loop Snooper Scoops the Halting Decider

If $Q$ loops on $Q$, $P$ says ‘Bad’ so $Q$ halts on $Q$!
The Loop Snooper Scoops the Halting Decider

If $Q$ halts on $Q$, $P$ says ‘Good’ so $Q$ loops on $Q$!
The Loop Snooper Scoops the Halting Decider

Whatever $P$ says $Q$ does, $Q$ does the opposite

![Diagram showing theloop snooper and halting decider interactions]

Can’t exist!
Your PC ran into a problem that it couldn't handle
Your PC ran into a problem that it couldn't handle

Scooping the Loop Snooper
Geoffrey K. Pullum
Professor Emeritus of General Linguistics
University of Edinburgh
Turing and Theoretical Computer Science

Church had already broken the Entscheidungsproblem a few months previously. Turing learned this, and went to Princeton to study with him.

Turing’s idea of computing is much more intuitive than Church’s – but they were able to show they were the same. Now both are part of the foundations of computer science.

Turing continued working on computation and logic until war broke out.
Turing at Bletchley Park

Turing joined the code-breaking team at Bletchley Park. He was instrumental in many of their greatest achievements, including the breaking of Enigma.

Reconstruction of the Bombe – Bletchley Museum
Turing and Computers

After the war, Turing worked in London and Manchester on the first real computers, as well as on mathematics.

The Manchester Mark 1
The Turing Test

He also thought about *Artificial Intelligence*, and proposed a test for truly intelligent computers:
Turing the biologist

He also studied how patterns (e.g. spots and stripes on fish) arise in biology.

He made clever mathematical models of chemical reactions to do this. Biologists still use his work.
Life and Death

Turing was homosexual, and in January 1952, after his house was burgled by an acquaintance of his boyfriend, he acknowledged this to the police. He was charged with gross indecency, and pled guilty. He was offered a choice between prison and an experimental hormone treatment intended ‘to reduce libido’. He took the treatment, for a year. Another year later, on 8 June 1954, he was found dead of cyanide poisoning – he used cyanide for chemistry and electroplating experiments. The inquest found suicide, and his biographers mostly agree. His relatives and closest friends thought it was accident.
Hilbert’s Tenth Problem

Solve *Diophantine equations*: polynomial equations about integers.
E.g., find $w, x, y, z$ such that $w^3 + x^3 = y^3 + z^3$.
(Many answers, e.g. 12, 1, 9, 10.)
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It turns out that this is *undecidable* – no program can solve it.

Took 40 years to prove: final step by Yuri Matiyasevich in 1970, building on previous steps by Martin Davis, Hilary Putnam and Julia Robinson.