Fixpoint alternation on trees

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1 Introduction

In his papers [Niw86,Niw97], Damian Niwiński studied fixpoint algebras over trees. In the associated fixpoint logic, formulae are built from n-ary function symbols, disjunction, and least and greatest fixpoint operators. The structures are infinite trees such that each node with n children is labelled with some n-ary function symbol; a node satisfies the formula $f(\phi_1, \ldots, \phi_n)$ if the node is labelled by f and the ith child satisfies ϕ_i; the fixpoints are taken over sets of nodes; a tree satisfies a formula if its root does. Niwiński established a number of results about such algebras, including intimate and now well-known relationships to automata theory. One result in particular concerns us here: he established a strict hierarchy of tree languages definable according to the number of alternating fixpoint quantifiers in the formula. In fact this hierarchy is closely connected to the hierarchy of Rabin indices in Rabin automata languages. However, in this study, as we have mentioned, the only boolean operator was union, and *not* intersection. Although intersection is easily added as a function symbol, many of the results rely on it not being a primitive of the language; and in particular, the alternation hierarchy theorem is not established for the languages with intersection. This problem has remained open.

A closely related fixpoint alternation problem is that in the modal mu-calculus. This problem also remained open for several years, and was solved independently by G. Lenzi [Len96] and myself [Bra96] in 1996. Since one can view the tree algebras as special cases of the modal mu-calculus, one thereby obtains a form of hierarchy for tree algebras with intersection. However, in both proofs the number of labels increases with the alternation depth, and so one does not obtain a *single* tree algebra with finite signature and intersection over which the fixpoint alternation hierarchy is strict—so to this extent Niwiński's problem remains open.

In this note, I show how to adjust and encode my proof of the modal mu-calculus alternation hierarchy in such a way as to establish a single tree algebra of finite signature with intersection on which the fixpoint alternation hierarchy is strict. In part 2, I discuss the strengthenings of the modal mu-calculus result required, and in part 3 apply the strengthened result to trees. I have tried to give in this short space the essentials of the proof as well as the result; obviously many details have been omitted. For background on tree algebras, refer to [Niw97], which is an excellent exposition.

2 Modal mu-calculus alternation on bounded-branching systems.

We consider the modal mu-calculus in the form with labelled modalities $\langle a\rangle\Phi$, and atomic propositions $P, Q, \ldots$. It is shown in [Bra98] that the formulae Hard_n, for each $n \geq 1$,

$$\mathrm{Hard}_n \stackrel{\mathrm{def}}{=} \mu X_n.\, \nu X_{n-1}.\, \ldots \mu\!\!\!/\nu X_1.\, (\langle c\rangle \mathrm{tt} \wedge [c]X_1) \vee \langle d\rangle X_1 \vee \langle x_1\rangle X_1 \vee \ldots \vee \langle x_n\rangle X_n$$

are strict Σ^μ_n (n alternating fixpoints, Σ outermost—$\mu\!\!\!/\nu$ is μ for odd n, ν for even n) formulae. This is shown by constructing a transition system $\mathscr{T}_n$ in which the set of states satisfying Hard_n has descriptive complexity strict Σ^μ_n in mu-arithmetic (first-order arithmetic with fixpoints); and any modal Σ^μ_r formula has at most Σ^μ_r arithmetic complexity.

There are two problems with this construction, if we wish to solve Niwiński's problem: firstly, the formula uses $n+2$ different labels, and secondly the c and d transitions are non-deterministic, and even infinitely branching. For reasons that will become apparent, we should like to have a transition system with exactly three labels, a_L, a_R, b such that every state has either one b successor or one a_L and one a_R successor.

To deal with these problems we build a new transition system $\mathscr{T}'_n$ in which the new labels code the old labels. One such coding is thus: if $s \xrightarrow{x_i} t$ in $\mathscr{T}_n$, then in $\mathscr{T}'_n$ we make $s \xrightarrow{b^{n+2}} ? \xrightarrow{a_L} t$, where the intermediate states are new 'junk' states, and also $? \xrightarrow{a_R}$ junk; if $s \xrightarrow{c} t_i$ for $1 \leq i \leq r$ (r is always either 2 or ∞), then we make:

$$\begin{array}{ccccccccc} s & \xrightarrow{b} & ? & \xrightarrow{a_L} & ? & \xrightarrow{a_L} & ? & \xrightarrow{a_L} & \cdots \\ & & \downarrow a_R & & \downarrow a_R & & \downarrow a_R & & \\ & & t_1 & & t_2 & & t_3 & & \end{array}$$

terminating with a string of junk states for the case $r = 2$; and if $s \xrightarrow{d} t_i$, we do the same, but with $s \xrightarrow{b^2}$ at the start. Thus we have used the number of b's to code the label of the original transition, and the a transitions to replace a single r-way branch by a sequence of r states.

We now need to translate the formula Hard_n for the new system. Obviously we can translate $\langle x_i\rangle$ into $\langle b\rangle^{i+2}\langle a_L\rangle$, but the translation of the other modalities requires introducing additional fixpoints. The (sometimes infinite) branching box $[c]X_1$ becomes $[b]\nu Y.\, [a_R]X_1 \wedge [a_L]Y$; and the branching diamond $\langle d\rangle X_1$ becomes $\langle b\rangle\langle b\rangle\mu Y.\, \langle a_R\rangle X_1 \vee \langle a_L\rangle Y$.

Since the construction of $\mathscr{T}'_n$ is recursive, the new formula Hard'_n still denotes an arithmetic Σ^μ_n-hard set of states. Unfortunately, Hard'_n is now a modal Σ^μ_{n+1} formula, owing to the introduction of the new fixpoints! This is still sufficient to establish the hierarchy, since it cannot be equivalent to any modal Π^μ_n formula. However, we can also obtain a direct proof that it cannot be equivalent to any modal Π^μ_{n+1} formula, since it can be shown, by extending an analysis of [Bra96a], that on a bounded degree recursive transition system a modal Σ^μ_n formula denotes a set of at most arithmetic Σ^μ_{n-1} complexity.

We now have the result

Theorem 1 *There is a class of transition systems with branching degree ≤ 2 on which the modal mu-calculus alternation hierarchy is strict.*

3 Alternation in tree algebras.

Owing to the way in which we have set up $\mathscr{T}_n'$, the transfer to tree algebras is almost immediate. We take the signature Σ with a binary function symbol a and a unary function symbol b. Thus a tree over this signature is a transition system of the form specified for $\mathscr{T}_n'$, and conversely the unwinding $\mathscr{T}_n''$ of $\mathscr{T}_n'$ is a tree over this signature. Since unwinding is a recursive operation, we can say that given a Σ_n^μ tree mu-formula, the set of nodes of $\mathscr{T}_n''$ satisfying it is of arithmetic complexity Σ_{n-1}^μ. However, we can translate Hard_n' into a tree formula Hard_n'':

$$\mu X_1 \ldots . \mu X_n . \, b(\nu Y. \, a(X_1, Y)) \vee b(b(\mu Y. \, a(X_1, \mathrm{tt}) \vee a(\mathrm{tt}, Y))) \vee b(b(b(X_1))) \ldots$$

such that the nodes of $\mathscr{T}_n''$ satisfying Hard_n'' are exactly those that are the unwindings of states of $\mathscr{T}_n'$ satisfying Hard_n'; and hence this set is arithmetic Σ_n^μ-hard. It follows that for any formula ϕ of lower alternation depth, there is a node on which ϕ and Hard_n'' disagree. Now by the crucial fact that a node satisfies a formula iff the subtree rooted at that node satisfies it (the 'internalization property' of [Niw97]), there is a tree on which ϕ and Hard_n'' disagree. Hence we have

Theorem 2 *The fixpoint alternation hierarchy over the tree logic with signature Σ and intersection is strict.*

One may note that our signature is a little more generous than required; we could take a nullary constant c instead of the unary b, as is done in [Niw97], at the price of slightly expanding the modalities. Thus the result applies for any signature with at least two symbols, at least one of which has arity at least 2, just as for the intersection-free hierarchy of [Niw97].

It is interesting to note that even though we are establishing a hierarchy with intersection, the hard formulae do not themselves use intersection; they need only the 'implicit' intersection given by the binary symbol a. However, unlike the hard formulae of [Niw97], which are in fact all equivalent to level Σ_2^μ formulae in the intersection-full hierarchy, the implicit conjunction conjoins different fixpoints. One may also compare the disjunctive formulae of [JaW95].

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