Fixpoints and Games

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Inductive definitions

Suppose Φ is an operator taking *sets* $W \subseteq \omega$ to sets $\Phi(W)$. If Φ is monotone, then

$$\varnothing \subseteq \Phi(\varnothing) \subseteq \Phi(\Phi(\varnothing)) \subseteq \ldots \textit{transfinitely}$$

The limit is the set *inductively defined by* Φ.
Put $\Phi^\zeta = \Phi(\bigcup_{\xi<\zeta} \Phi^\xi)$; then $\Phi^\infty = \bigcup_\zeta \Phi^\zeta$ is the limit.

If Φ is definable as $W \mapsto \{\, w \mid \phi(W, w) \,\}$
for a Γ formula $\phi(W, w)$ with a free set variable W, then write $\phi^\infty(w)$ for $w \in \Phi^\infty$.

If ϕ also has free variable x, and $w_0 \in \omega$, can define

$$R(x) \Leftrightarrow \phi^\infty(x, w_0)$$

Such an R is said to be *Γ-IND*.
If ϕ is *positive*, then R is *pos-Γ-IND*.

Kleene showed that

$$\textit{pos-}\Pi^0_1\textit{-IND} = \Pi^1_1$$

The fixpoint hierarchy

Take *first-order* arithmetic and add set variables X, membership $\in$, and an operator to form inductive definitions:

$$\mu(w, W).\phi(w, W)$$

denotes the set inductively defined by ϕ (must be positive).

Define $\Sigma^\mu_0 = \Sigma^0_1$; $\Pi^\mu_n = \neg\Sigma^\mu_n$;
and $P(x)$ to be Σ^μ_{n+1} if there is Π^μ_n $Q(w, W, x)$ such that
$P(x) \Leftrightarrow \tau \in \mu(w, W).Q(w, W, x)$.

Note that $\Sigma^\mu_1 = \textit{pos-}\Pi^0_1\textit{-IND} = \Pi^1_1$.

The fixpoint hierarchy is strict by 'the usual argument'.

. . . came from Computer Science

In modal logic, don't need individual variables: $[]\Phi$ means $s \in []\Phi$ where s is the 'current state'.

So the fixpoint extension of modal logic looks like $\mu Z.\Phi(Z)$, with Z a variable over sets of states.

The fixpoint hierarchy in modal fixpoint logic (alias modal mu-calculus) $\mu Z_1.\nu Z_2.\mu Z_3 \ldots$ is important for several reasons – and was early understood to be intimately connected with *Rabin automata* and *parity games*.

Automata and modal mu-calculus

A *Rabin automaton* is a finite automaton equipped with m pairs (R_i, G_i) of subsets of states.

An infinite run is *accepted* if there some i such that R_i is seen finitely often and G_i is seen infinitely often: or, so to say,

$$\bigvee_{1 \leq i \leq m} (\neg \infty R_i \wedge \infty G_i)$$

Rabin automata correspond to certain fixpoint languages; m corresponds to alternation depth.

In an *alternating* Rabin automaton, we play a Gale–Stewart style game on the automaton, with the above winning condition.

They are equivalent to *alternating parity automata*. Here we have sets $X_1, X_2, X_3, X_4, \ldots, X_{2m}$; the winning condition is: the highest X_i seen infinitely often must be even.

These are equivalent to modal mu-calculus: $2m$ corresponds to fixpoint alternation.

Now, the parity condition says:

$$\infty X_{2m} \vee (\neg\infty X_{2m-1} \wedge (\infty X_{2m-2} \\ \vee \ldots \wedge (\infty X_2 \vee \neg\infty X_1) \ldots))$$

Parity and Rabin conditions are boolean combinations of ∞ and $\neg\infty$.

The game quantifier; determinacy

Let $P(\alpha, x)$ define a family of games; define

$$\Game\alpha.P(\alpha, x) \Leftrightarrow x \in \{\, x \mid \text{Eloise wins } P(\alpha, x) \,\}$$

(so loosely $\Game = \exists\forall\exists\forall \ldots$)
If P is Γ then $\Game\alpha.P$ is $\Game\Gamma$.

Martin's theorem says $Det(\mathbf{\Delta}^1_1)$: if P is $\mathbf{\Delta}^1_1$ then the game P is determined.

If Γ is a known class, what do we know about $\Game\Gamma$?

In general: if $Det(\boldsymbol{\Gamma})$ (and $\ldots$) then $\neg\Game\Gamma = \Game\neg\Gamma$.

For analytical Γ we have $\Game\Pi^1_n = \Sigma^1_{n+1}$; and assuming $Det(\mathbf{\Sigma}^1_n)$ also $\Game\Sigma^1_n = \Pi^1_{n+1}$.

Games and induction

Kechris & Moschovakis showed that

$$\Game\Sigma^0_1 = \Pi^1_1$$

Solovay showed

$$\Game\Sigma^0_2 = \textit{pos-}\Sigma^1_1\textit{-IND}$$

Now $\Sigma^1_1 = \neg\Pi^1_1 = \neg\Sigma^\mu_1$. So putting it another way:

$$\begin{aligned} \Game\Sigma^0_1 &= \Sigma^\mu_1 \\ \Game\Sigma^0_2 &= \Sigma^\mu_2 \end{aligned}$$

The computer science version suggests how to continue ...

Playing games with arithmetic fixpoints

Take an arithmetic fixpoint formula of the form

$$\mu X_{2m-1}.\nu X_{2m-2}.\ldots.\nu X_2.\mu X_1.\phi$$

We can define a game P on ω such that Eloise wins iff the formula is true.

How? It is exactly an 'interpreter': build interpreter machine, with states coded as integers. For correctness, need exactly that the plays satisfy a parity condition: highest X_i seen infinitely often is even. (Why? By transferring via modal mu-calculus; or directly.)

Therefore *any* fixpoint property Q is $\Game\alpha.P$ for some parity condition P.

What is a parity condition? '∞X_i in α' says that $\forall j.\exists k > j.$'X_i seen at $\alpha(k)$'.
So it is Π^0_2; so parity is ∇^0_2 (boolean closure of Σ^0_2).

From fixpoints via parity to difference hierarchies

So we have $\Sigma^{\mu}_{n} \subseteq \Game\nabla^{0}_{2}$ for all n.

Can we refine this?

The *difference hierarchy over* Σ^{0}_{2} is defined by

$$\Sigma^{\partial}_{1} = \Sigma^{0}_{2} \qquad \Sigma^{\partial}_{n+1} = \Sigma^{0}_{2} \wedge \Pi^{\partial}_{n}$$

We may as well define $\Sigma^{\partial}_{0} = \Sigma^{0}_{1}$. Then Kechris–Moschovakis and Solovay give us

$$\Sigma^{\mu}_{1} = \Game\Sigma^{\partial}_{0} \qquad \Sigma^{\mu}_{2} = \Game\Sigma^{\partial}_{1}$$

Is this the right formulation for the generalization?

By inspection the parity condition of rank m is Σ^{∂}_{m}.

By more careful inspection it is actually Σ^{∂}_{m-1}.

(Why? Because rank 1 is 'finitely often X_1', which can only be true if play terminates: $\exists j.$'play stops at $\alpha(j)$' which is only Σ^{0}_{1}.)

From differences to fixpoints

So we now have

$$\Sigma^{\mu}_{n} \subseteq \text{Ↄ}\Sigma^{\partial}_{n-1}$$

The converse is harder. The idea is to extend Solovay, which analysed Wolfe's proof of $Det(\mathbf{\Sigma}^0_2)$.

Suppose $P(a) \Leftrightarrow (\exists i. Q(i,\alpha)) \wedge R(\alpha)$ is Σ^{∂}_{n+1},
so Q is Π^0_1 and R is Π^{∂}_{n}.

We define inductively 'easy winning positions' $u = a_0 \dots a_k$ for Eloise by

$W^{\zeta} = \{\, u \mid \exists i.\text{'Eloise wins } H^{\zeta}_{i} \text{ from } u\text{'}\,\}$,

where H^{ζ}_{i} is defined in terms of R, $W^{<\zeta}$ and Q.

It can be shown that Eloise wins from u iff $u \in W^{\infty}$. Then W is an inductive definition with a (by induction) Π^{μ}_{n+1} body, so is Σ^{μ}_{n+2}, Q.E.D.

So we have the theorem

$$\Sigma^{\mu}_{n+1} = \text{Ↄ}\Sigma^{\partial}_{n}.$$

Onwards and Upwards . . .

It is 'well-known' that $\Delta^0_3 = \bigcup_{\zeta < \omega_1^{CK}} \Sigma^{\partial}_{\zeta}$ (and similarly for higher Δ^0_n), for the transfinite difference hierarchy.

So is it the case that

$$\Sigma^{\mu}_{\zeta+1} = \Game\Sigma^{\partial}_{\zeta}?$$

(For $\zeta < \omega_1^{CK}$ and appropriate transfinite extension of fixpoint hierarchy.)

Yes . . . though now fixpoint satisfaction no longer corresponds to general transfinite parity games, but rather to certain well-behaved transfinite parity games.

(Joint work with Jacques Duparc and Sandra Quickert)

. . . with WWW

Another way of understanding what happens as we climb the difference hierarchy.

The effective Wadge degrees of Σ_n^{∂} are $(\omega_1^{\mathrm{CK}})^n$ (and we hope this continues transfinitely).

There is an operation on Wadge games which has the effect of increasing the degree by a multiplicative factor of ω_1^{CK}.

This operation is roughly: allow one player to cancel the game and start again, perhaps switching to the complement of 'his' domain. He's allowed to do this a finite number of times.

In a loose sense, this corresponds at the strategy computation level to wrapping an inductive definition around your existing strategy.

Question: what does the game quantifier do to Wadge degrees?

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